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Cohomology, Central Extensions, and (Dynamical) Groups

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We analyze in this paper the process of group contraction which allows the transition from the Einstenian quantum dynamics to the Galilean one in terms of the cohomology of the Poincaré and Galilei groups. It is shown that the cohomological constructions on both groups do not commute with the contraction process. As a result, the extension coboundaries of the Poincaré group which lead to extension cocycles of the Galilei group in the "nonrelativistic" limit are characterized geometrically. Finally, the above results are applied to a quantization procedure based on a group manifold.

1. INTRODUCTION

The relevance of the $U(1)$-central extensions $\tilde{G}_{(m)}$ of the ten-parameter Galilei group $G(G=\tilde{G}_{(m)}/U(1))$ in "nonrelativistic" quantum mechanics is well known since the work of Bargmann[1] (1954). The possibility of having nontrivial extensions is tied to the nontrivial cohomology of the Galilei group, a fact which has recently been exploited (Aldaya and de Azcárraga, 1982, 1984; Aldaya et al., 1984) to define a group theoretical approach to geometric quantization based on the $U(1)$ principal fibered structure $P(M, U(1))$ which can be defined for the extended Galilei group $[\tilde{G}_{(m)}(G, U(1))]$ or "quantum group." As is known since the early work of Wigner (1939), the situation for the (relativistic) Poincaré group is completely different. The fact that the Poincaré group has trivial cohomology is usually translated by stating that the projective unitary representations of the Poincaré group $\mathscr{P}=\mathrm{Tr}\circ L_+^\uparrow$ ($\circ$ means semidirect product) are in fact obtained from the unitary representations of its universal covering group $\bar{\mathscr{P}}=\mathrm{Tr}\circ SL(2, C)$ and so the phase factors are reduced to a $\pm$ sign. $\bar{\mathscr{P}}$ was called by Wigner (1964) the "quantum mechanical group."

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The Galilei and Poincaré relativity groups, relevant in the nonrelativistic and relativistic dynamics, respectively, are groups related by contraction (Inönü and Wigner, 1953; Inönü, 1964), a process which is realized as the limit $c \to \infty$. However, the structure of $\tilde{G}_{(m)}$—central extension of G by $U(1)$—and that of $P \otimes U(1)$ are completely different, and the question arises of knowing when a coboundary generating a necessarily direct product extension of $\mathcal{P}$ leads to a true cocycle of G. This question was answered first by Saletan (Saletan, 1961; Hermann, 1966) in algebraic terms. Here we wish to reexamine the problem in differential geometric terms, and to analyze the noncommutativity of the contraction process in connection with the canonical construction of a quantum dynamical structure on the group of symmetry. This study is interesting because it constitutes a possible way out to the problem of formulating a relativistic quantum dynamics on a group manifold which in a first stage would be forbidden since the trivial Poincaré group cohomology prevents its extension by $U(1)$. More precisely, the following analysis establishes the difference between a purely relativistic quantum dynamics—associated with the Poincaré *group*—which leads through contraction to the corresponding Galilean dynamics, and the customary relativistic dynamics which is based on an *action* of the Poincaré group—as realized by the customary quantum operators—but whose associated equations of motion (as, for instance, the Klein-Gordon equation) require the previous substraction of the rest mass energy mc^2 in order to produce a nonrelativistic limit. To this aim, we analyze (Section 2) the characterization of $U(1)$ central extensions $\tilde{G}$ of a group G and how a direct product extension $G \otimes U(1)$ may lead through contraction to a nontrivial one $\tilde{G}_c$ of the contracted group G_c. We also study how the selection of a coboundary modifies the geometric differential structure which can be built on the extended group and, in particular, how an invariant connection can be defined on the fibered structure $\tilde{G}(G, U(1))$. Section 3 is devoted to considering the Poincaré and Galilei cases, showing explicitly the different connection forms (flat and nonflat) which may be defined on $\mathcal{P} \otimes U(1)$ $(\mathcal{P}, U(1))$. Finally, in Section 4 we discuss the application of the above constructions to the formulation of a quantum dynamics.

2. CENTRAL EXTENSIONS, COHOMOLOGY, AND THE CONTRACTION PROCESS

Let G be a Lie group and let $\tilde{G}$ be a central extension of G by $U(1)$ with composition law

$$\tilde{g}'' = \tilde{g}' * \tilde{g} = (g' * g, \zeta'\zeta \exp i\xi(g', g)) \equiv (g'', \zeta'') \tag{1}$$

were $\zeta \equiv e^{i\theta}$ and $\xi\colon G\times G\to\mathbb{R}$ is the two-cocycle of the extension which satisfies (Bargmann, 1954)

$$\begin{aligned} &\xi(e,g)=0, \qquad \xi(g',g)+\xi(g'g,g'')=\xi(g,g'')+\xi(g',gg'') \\ &\xi(g,e)=0 \end{aligned} \tag{2}$$

If we perform the reparametrization

$$\zeta\mapsto\zeta_0=\zeta\exp i\delta(g), \qquad g\mapsto g \tag{3}$$

where δ is a real function on G, $\delta(e)=0$, (2) will now be written as

$$\tilde{g}'*\tilde{g}\equiv(g',\zeta_0')*(g,\zeta_0)=(g'g,\zeta_0'\zeta_0\exp i\xi_0(g',g))\equiv(g'',\zeta_0'') \tag{4}$$

Comparing ζ_0'' with ζ'' one immediately obtains

$$\xi_0(g',g)=\xi(g',g)+\delta(g'g)-\delta(g')-\delta(g) \tag{5}$$

A cocycle of the form $\xi(g',g)=\delta(g'g)-\delta(g')-\delta(g)$ is called a coboundary; for such a ξ, the law (1) is in fact the direct product law as we may use (5) to get $\xi_0(g',g)=0$. The extensions are thus characterized by the second cohomology group $H_0^2(G,U(1))$ (i.e., cocycles/coboundaries).

Let us now consider in which way the addition of a coboundary modifies the expression of the Lie algebra generators of (a trivial extension) $\tilde{G}=G\otimes U(1)$ as given, say, by the left invariant vector fields $\tilde{X}^L$ on $\tilde{G}$. We shall assume, moreover, that their original form has been obtained for the case in which $\xi=0$, i.e., that they are just given by the $U(1)$ vector field $\Xi=i\zeta(\partial/\partial\zeta)$ plus the $i=1\cdots r$ vector fields $X^L_{g_i}$ on G (the g_i parametrize the elements of G). Under the addition of a coboundary we get from (5) with $\xi=0$

$$\begin{aligned} \tilde{X}^L_{g_i}&=X^L_{g_i}+\left.\frac{\partial\xi_0(g',g)}{\partial g_i}\right|_{g=e}\Xi \\ &=X^L_{g_i}+\frac{\partial}{\partial g_i}[\delta(g'g)-\delta(g')-\delta(g)]|_{g=e}\,\Xi \\ &=X^L_{g_i}+\left[L_{X_{g_i}}\delta(g)-\left[\frac{\partial\delta(g)}{\partial g_i}\right]_{g=e}\right]\Xi\equiv X^L_{g_i}+X^{L\zeta}_{g_i}\Xi \end{aligned} \tag{6a}$$

where $L_{X_{g_i}}$ is the Lie derivative, plus

$$\tilde{X}^L_\zeta=\Xi=i\zeta\frac{\partial}{\partial\zeta} \qquad \left(\frac{\partial}{\partial\theta};\ \zeta=e^{i\theta}\in U(1)\right) \tag{6b}$$

(6a) shows that the $\tilde{X}^L_{g_i}$ (initially just $X^L_{g_i}$ in the zero coboundary parametrization) will in general acquire a $U(1)$ component. Because of (6a) and the fact that $\partial\delta(g)/\partial g_i|_{g=e}$ is a constant, the commutator of two elements $\tilde{X}^L_{g_i} \equiv \tilde{X}$, $\tilde{X}^L_{g_j} \equiv \tilde{Y}$ of the algebra (we shall omit the superscript L henceforth) is now given by

$$\begin{aligned}[\tilde{X}, \tilde{Y}] &= [X, Y] + ([X, Y] \cdot \delta(g))\Xi \\ &= Z + (L_Z\delta(g))\Xi = \tilde{Z} + \left[\frac{\partial\delta(g)}{\partial g_Z}\right]_{g=e} \Xi\end{aligned} \tag{7}$$

The last term may be absent; in this case, there is no modification of the structure constants.

In the same way we may now consider in which form the canonical left-invariant form θ on the group, and more specifically, the component Θ dual to the $U(1)$ generator Ξ, is modified by the addition of a coboundary. Such a 1-form is defined by the conditions

$$\Theta(\tilde{X}_{g_i}) = 0 \quad \forall \tilde{X}_{g_i}, \qquad \Theta(\Xi) = 1$$

When $\delta = 0$, $\tilde{X}_{g_i} = X_{g_i}$ and the $U(1)$ component of the canonical 1-form on $G \otimes U(1)$ is given by $d\zeta/i\zeta$. When the coboundary is nonzero, it modifies the form of the invariant vector fields and the new expression for the canonical $U(1)$ component is given in general by

$$\Theta = \Theta_{g_j}\, dg^j + \frac{d\zeta}{i\zeta} \qquad (\Theta_\zeta = 1) \tag{8}$$

Writing the vector fields on $G \otimes U(1)$ as

$$\tilde{X}_{(g_i)} = X^j_{(g_i)} \frac{\partial}{\partial g_j} + X^\zeta_{(g_i)}\Xi \tag{9}$$

where $X^\zeta_{(g_i)} = L_{X_{(g_i)}}\delta(g) - [\partial\delta(g)/\partial g_i]_{g=e}$ [(6a)], the components of Θ satisfy

$$\Theta_{g_j} X^j_{(g_i)} + X^\zeta_{(g_i)} = 0 \tag{10}$$

One checks that, since all $X_{(g_i)}$ are independent, $\Theta_{g_j} = 0$ when $X^\zeta_{(g_i)} = 0$, and in this case $(\delta(g) = 0)\Theta$ reduces to $d\zeta/i\zeta$. When the terms $X^\zeta_{(g_i)}$ in Ξ are of the form $L_{X_{(g_i)}}\delta(g)$, i.e., when $\partial\delta(g)/\partial g_i|_{g=e} = 0$, (10) reads

$$\Theta_{g_j} X^j_{(g_i)} + X^j_{(g_i)} \frac{\partial\delta(g)}{\partial g_j} = 0, \qquad \Theta_{g_j} = -\frac{\partial\delta(g)}{\partial g_j} \tag{11}$$

which implies

$$\Theta = \frac{d\zeta}{i\zeta} - d(\delta(g)) \tag{11'}$$

where d is the exterior derivative and so $d\Theta = 0$. When $X^{\zeta}_{(gi)}$ is given by its full form (6a), $d\Theta$ defines a presymplectic form as in the case of a true extension. In either case, the change of variables (3) brings Θ to the form $d\zeta / i\zeta$ as corresponds to a direct product extension.

Because of the $U(1)$-principal fibered structure, it is clear that Θ may be understood (see Section 3) as a connection form defining the horizontal components of vector fields: $\tilde{X}$ is horizontal iff $\Theta(\tilde{X}) = 0$. Thus, the vector fields $\tilde{X}_{(gi)}$ are horizontal (despite the Ξ component) and Ξ is vertical. Nevertheless only when $\tilde{X} = X + (L_X \delta(g))\Xi$ [cf. (6a)] the horizontal vector fields generate a subalgebra (isomorphic to the algebra $\mathscr{G}$ of G).

We may now discuss the contraction process in the above cohomological context. In it, the question of obtaining a nontrivial central extension $\tilde{G}_c$ of the contracted group G_c from a trivial (direct product) one of a group G is a consequence of the noncommutativity of the following diagram:

$$\begin{array}{ccc} G\otimes U(1) & \xrightarrow{\text{contraction}} & G_c \otimes U(1) \\ \Big\downarrow \begin{array}{l}\scriptstyle\text{change of}\\ \scriptstyle\text{coboundary}\end{array} & & \not\Vert \\ G\otimes U(1) & \xrightarrow{\text{contraction}} & \tilde{G}_c \end{array}$$

where in the first line we have assumed that the direct product $G\otimes U(1)$ is directly given by the usual direct product law (no coboundary). Thus, we have to determine the class of *coboundaries* of G which in the contraction process give rise to true *cocycles* of $G_c = \tilde{G}_c / U(1)$. (In the particular case we shall be interested in, $G = \mathscr{P}$ (Poincaré), $G_c = G$ (Galilei), $\tilde{G}_c = \tilde{G}_{(m)}$ [$U(1)$-extended Galilei group].) Because of (7), when

$$\left.\frac{\partial \delta(g)}{\partial gi}\right|_{g=e} = 0 \tag{12}$$

the commutation relations for the vector fields $\tilde{X}$ of $G\otimes U(1)$ are explicitly those of the direct product, and the contraction process cannot change this situation: we are in the first line of the above diagram. But if (12) is not zero we may obtain a contracted group $\tilde{G}_c$ which will be a central extension of G_c trivial or not according to whether the contracted $\delta(g)$ is well defined or not. Obviously if $\delta(g)$ is not defined in the contraction limit, the generated coboundary $\xi(g', g) = \delta(g' * g) - \delta(g') - \delta(g)$ will lead, if defined in such a limit, to a true cocycle of the contracted group (Saletan, 1961; Hermann, 1966) since there will be no generating function for such a cocycle.[2]

3. THE CASE OF THE POINCARÉ AND GALILEI GROUPS. LEFT-INVARIANT VECTOR FIELDS AND PRINCIPAL BUNDLE STRUCTURE WITH CONNECTION

Because the function $\delta: G \to \mathbb{R}$ is an exponent, it is clear that $\delta(g)$ has to be dimensionless. [We use exponents of the form $\exp[(i/\hbar)\xi]$ and put $\hbar = 1$ throughout; one may notice at this stage the quantum character of the $U(1)$ extensions because a constant with dimensions of action is needed in the exponential to give a dimensionless exponent.] Let us parametrize the Poincaré group elements by $(A^0, \mathbf{A}, \boldsymbol{\alpha}, \boldsymbol{\varepsilon})$ where $(A^0, \mathbf{A})$ are the time-space translations and

$$\boldsymbol{\alpha} = \frac{\gamma}{[2(1+\gamma)]^{1/2}} \frac{\mathbf{V}}{c}, \qquad \gamma = \frac{1}{(1 - V^2/c^2)^{1/2}}, \qquad |\boldsymbol{\varepsilon}| = \sin\frac{\varphi}{2} \tag{13}$$

characterize the boosts and rotations (of angle φ around the axis $\boldsymbol{\eta} = \boldsymbol{\varepsilon}/|\boldsymbol{\varepsilon}|$). It is also useful to use the momentum of a boosted particle of mass m to parametrize the Lorentz transformations

$$\mathbf{p} = 2mc\boldsymbol{\alpha}(1+\alpha^2)^{1/2} \qquad [p^0 = mc(1+2\alpha^2)]$$

In order to have (12) nonzero, it is clear that it has to contain a term linear in the group parameters and with dimensions of action (unity) and this can only be a term in which one factor is mc (or p^0) and the other a translation. Of these δ's, only those having a term of the form mcA^0 (or p^0A^0) will

[2]In this reasoning we have omitted any explicit indication of the dependence on the contraction parameter λ; in fact, $\delta(g)$ should be written $\delta(g, \lambda)$ etc. It is not difficult to see which mechanism associates coboundaries of the original group with coboundaries and cocycles of the contracted group. The contraction process is a homomorphism of the vector space of the Lie algebra, but obviously not of the Lie algebra structure, which gets "abelianized." However, given a cocycle of the contracted algebra $\mathcal{G}_c$ of G_c, i.e., a two-form

$$\Omega_c: \mathcal{G}_c \times \mathcal{G}_c \to \mathbb{R}/\Omega_c(Z_c, [Z'_c, Z''_c]) + \Omega_c(Z'_c, [Z''_c, Z_c]) + \Omega_c(Z''_c, [Z_c, Z'_c]) = 0$$

the homomorphism of vector spaces $f: \mathcal{G} \to \mathcal{G}_c$ induces a bilineal form $\Omega \equiv f^*\Omega_c: \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ in the natural way. If Ω has to be a cocycle of $\mathcal{G}$, it has to satisfy

$$\Omega(Z, [Z', Z'']) + \Omega(Z', [Z'', Z]) + \Omega(Z'', [Z, Z']) = 0$$

Despite the fact that, in general, $[Z_c, Z'_c] \neq [Z, Z']_c$ the above expression will be true if, when both commutators are different, the difference belongs to ker Ω. This is the case for the Poincaré and Galilei groups, for which (see Section 3) $f([X_{A^i}, X_{\alpha j}]) = 2\delta_{ij} X_{A^0} \neq [f(X_{A^i}), f(X_{\alpha j})] = 0$ and $X_{A^0} \in \text{Ker}\,\Omega$.

generate a true cocycle in the nonrelativistic limit [$\delta(g)$'s such as mcA^1 will generate a coboundary with no "nonrelativistic" limit]. We may thus define an equivalence relation within the Poincaré coboundaries ξ with nonrelativistic ξ_c limits by identifying $\delta(g)$ such that $\delta(g)\to\infty$ for $c\to\infty$ and such that $\partial(\delta(g)-\delta'(g))/\partial g|_{g=e}=0$ (for instance, mcA^0 and p^0A^0 belong to the same "class"). The resulting "cohomology group" is isomorphic to the (true) Galilei cohomology group, which, as is well known (Bargmann, 1954), is simply parametrized by the mass.

As a first example, let us evaluate the left-invariant vector fields for the coboundary generated by $\delta(g)=-mcA^0$ which is given by

$$\xi(g',g)=-2mc\alpha'^2A^0-2mc(1+\alpha)^{1/2}\boldsymbol{\alpha}'\cdot R'\mathbf{A} \tag{14}$$

With the parametrization (13), the Poincaré left-invariant vector field are

$$X^L_{(A^0)}=(1+2\alpha^2)\frac{\partial}{\partial A^0}+2(1+\alpha^2)^{1/2}\boldsymbol{\alpha}\cdot\frac{\partial}{\partial \mathbf{A}}$$

$$X^L_{(A^k)}=2(1+\alpha^2)^{1/2}R^j_k\alpha_j\frac{\partial}{\partial A^0}+(R^i_k+2\alpha^i\alpha_jR^j_k)\frac{\partial}{\partial A^i}$$

$$X^L_{(\varepsilon^i)}=((1-\varepsilon^2)^{1/2}\delta^j_i+\eta^j_{\cdot ik}\varepsilon^k)\frac{\partial}{\partial\varepsilon^j}$$

$$X^L_{(\alpha^i)}=\frac{1}{(1+\alpha^2)^{1/2}}[R^j_i+\alpha_lR^l_i\alpha^j]\frac{\partial}{\partial\alpha^j}+\frac{1}{(1+\alpha^2)^{1/2}}[(1-\varepsilon^2)^{1/2}\eta^j_{\cdot ki}$$

$$-\eta^j_{\cdot sn}\eta^n_{\cdot ki}\varepsilon^s]R^{-1k}_m\alpha^m\frac{\partial}{\partial\varepsilon j} \tag{15}$$

For the direct product extension $\mathscr{P}\otimes U(1)$ as given by the coboundary (14) we get either by direct computation from the group law (1) or by using (6a)

$$\begin{aligned}
\tilde{X}^L_{(A^0)}&=X^L_{(A^0)}-2mc\alpha^2\Xi\\
\tilde{X}^L_{(A^k)}&=X^L_{(A^k)}-2mc(1+\alpha^2)^{1/2}\alpha_jR^j_k\Xi\\
\tilde{X}^L_{(\varepsilon^i)}&=X^L_{(\varepsilon^i)}\\
\tilde{X}^L_{(\alpha^i)}&=X^L_{(\alpha^i)}\\
\tilde{X}^L_{(\zeta)}&\equiv\Xi=i\zeta\frac{\partial}{\partial\zeta}
\end{aligned} \tag{16}$$

whose commutation relations are those of $\mathscr{P}$ (plus [Ξ, any $\tilde{X}$] = 0) but for the commutator

$$[\tilde{X}^L_{(\alpha^i)},\tilde{X}^L_{(A^k)}]=2\delta_{ik}(\tilde{X}^L_{(A^0)}-mc\Xi) \tag{17}$$

which also follows from (7).

The canonical left-invariant 1-form θ on the Poincaré group is derived from (15). The different components are

$$\begin{aligned}
\theta^L_{A^0} &= (1+2\alpha^2)\, dA^0 - 2(1+\alpha^2)^{1/2}\boldsymbol{\alpha}\cdot d\mathbf{A} \\
\theta^L_{\mathbf{A}} &= -2(1+\alpha^2)^{1/2}R^{-1}(\boldsymbol{\varepsilon})\boldsymbol{\alpha}\, dA^0 + R^{-1}(\boldsymbol{\varepsilon})(I+2\boldsymbol{\alpha}\otimes\boldsymbol{\alpha})\, d\mathbf{A} \\
\theta^L_{\boldsymbol{\varepsilon}} &= (1-\varepsilon^2)^{1/2} d\boldsymbol{\varepsilon} - (\boldsymbol{\varepsilon}\cdot d\boldsymbol{\varepsilon})(1-\varepsilon^2)^{-1/2}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\times d\boldsymbol{\varepsilon} - R^{-1}(\boldsymbol{\varepsilon})(\boldsymbol{\alpha}\times d\boldsymbol{\alpha}) \\
\theta^L_{\boldsymbol{\alpha}} &= R^{-1}(\boldsymbol{\varepsilon})[(1+\alpha^2)^{1/2} d\boldsymbol{\alpha} - (\boldsymbol{\alpha}\cdot d\boldsymbol{\alpha})(1+\alpha^2)^{-1/2}\boldsymbol{\alpha}]
\end{aligned} \tag{18}$$

The component $\theta^L_\zeta \equiv \Theta$ in Ξ of the direct product extension as given by the coboundary (14) is now readily found to be

$$\begin{aligned}
\Theta &= -2mc\alpha^2\, dA^0 + 2mc(1+\alpha^2)^{1/2}\boldsymbol{\alpha}\cdot d\mathbf{A} + \frac{d\zeta}{i\zeta} \\
&= \frac{d\zeta}{i\zeta} + \mathbf{p}\cdot d\mathbf{A} - (p^0 - mc)\, dA^0 = \frac{d\zeta}{i\zeta} - mc(\theta^L_{A^0} - dA^0)
\end{aligned} \tag{19}$$

instead of $\Theta = d\zeta/i\zeta$.

Let us now give the explicit formulas for the nonrelativistic limit. In it, the Poincaré coboundary (14) gives the Galilean cocycle

$$\xi(g', g) = -m(\tfrac{1}{2}\mathbf{V}'^2 B + \mathbf{V}'\cdot R'\mathbf{A}) \tag{20}$$

where $B \equiv A^0/c$, the left vector fields become those of the extended Galilei group

$$\begin{aligned}
\tilde{X}^L_{(B)} &= X^L_{(B)} - \tfrac{1}{2}mV^2\Xi = \frac{\partial}{\partial B} + \mathbf{V}\cdot\frac{\partial}{\partial \mathbf{A}} - \tfrac{1}{2}mV^2\Xi \\
\tilde{X}^L_{(A^i)} &= X^L_{(A^i)} - mR^k_i V_k \Xi = R^k_i\left(\frac{\partial}{\partial A^k} - mV_k\Xi\right) \\
\tilde{X}^L_{(\varepsilon^i)} &= X^L_{(\varepsilon^i)} = ((1-\varepsilon^2)^{1/2}\delta^j_i + \eta^j{}_{\cdot ik}\varepsilon^k)\frac{\partial}{\partial \varepsilon j} \\
\tilde{X}^L_{(V^i)} &= X^L_{(V^i)} = R^k_i \frac{\partial}{\partial V^k} \\
\tilde{X}^L_{(\zeta)} &\equiv \Xi
\end{aligned} \tag{21}$$

and the canonical form on $\tilde{G}_{(m)}$ is now found to be

$$\begin{aligned}
\theta^B &= dB \\
\theta^{\mathbf{A}} &= R^{-1}(\boldsymbol{\varepsilon})(d\mathbf{A}-\mathbf{V}\,dB) \\
\theta^{\boldsymbol{\varepsilon}} &= (1-\varepsilon^2)^{1/2}\,d\boldsymbol{\varepsilon} - d((1-\varepsilon^2)^{1/2})\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\times d\boldsymbol{\varepsilon} \\
\theta^{\mathbf{V}} &= R^{-1}(\boldsymbol{\varepsilon})d\mathbf{V} \\
\Theta &= m\mathbf{V}\cdot d\mathbf{A} - \tfrac{1}{2}m\mathbf{V}^2\,dB + \frac{d\zeta}{i\zeta}
\end{aligned} \tag{22}$$

We now give two more examples. For $\delta(g) = -A^\mu p_\mu$ we get the following Poincaré coboundary:

$$\xi(g', g) = -A'^\mu(\Lambda' p - p')_\mu, \qquad g', g \in P \tag{23}$$

whose limit is the Galilean cocycle

$$\xi(g', g) = m[\mathbf{A}'\cdot R'\mathbf{V} - B'(\tfrac{1}{2}V^2 + \mathbf{V}'\cdot R'\mathbf{V})], \qquad g', g \in G \tag{24}$$

Similarly, for $\delta(g) = 2mc(1+\alpha^2)^{1/2}\boldsymbol{\alpha}\cdot\mathbf{A} - mcA^0 = \mathbf{A}\cdot\mathbf{p} - mcA^0$ we get the following:

Poincaré coboundary,

$$\xi(g', g) = \left[\frac{\mathbf{p}'}{cm}A^0 + R'\mathbf{A} + \frac{(\mathbf{p}'\cdot R'\mathbf{A})}{mc(p_0'+mc)}\mathbf{p}'\right]\cdot\left[\gamma\mathbf{p}' + R'\mathbf{p} + \frac{(\mathbf{p}'\cdot R'\mathbf{p})}{mc(p_0'+mc)}\mathbf{p}'\right]$$
$$-(p_0' - mc)A^0 - \mathbf{p}'\cdot R'\mathbf{A} \tag{25}$$

Left-invariant fields;

$$\begin{aligned}
\tilde{X}^L_{(A^0)} &= X^L_{(A^0)} + 2mc\alpha^2(1+2\alpha^2)\Xi \\
\tilde{X}^L_{(A^k)} &= X^L_{(A^k)} + 2mc(1+\alpha^2)^{1/2}2\alpha^2\alpha_j R^j_k\Xi \\
\tilde{X}^L_{(\varepsilon^i)} &= X^L_{(\varepsilon^i)} \\
\tilde{X}^L_{(\alpha^i)} &= X^L_{(\alpha^i)} + 2mcA_m[R^m_i + 2\alpha^m\alpha_j R^j_i]\Xi \\
\tilde{X}^L_{(\zeta)} &= \Xi
\end{aligned} \tag{26}$$

Poincaré Θ,

$$\Theta = -\mathbf{A}\cdot d\mathbf{p} - (\mathrm{p}^0 - mc)\,dA^0 + \frac{d\zeta}{i\zeta} \tag{27}$$

Galilean cocycle,

$$\xi(g', g) = m[\mathbf{A}'\cdot R'\mathbf{V} + B(\mathbf{V}'\cdot R'\mathbf{V} + \tfrac{1}{2}\mathbf{V}'^2)] \tag{28}$$

Extended Galilei vector fields,

$$\tilde{X}^L_{(B)} = \frac{\partial}{\partial B} + \mathbf{V}\cdot\frac{\partial}{\partial \mathbf{A}} + \tfrac{1}{2}m\mathbf{V}^2\Xi$$

$$\tilde{X}^L_{(A^i)} = R^k_i \frac{\partial}{\partial A^k}$$

$$\tilde{X}^L_{(\varepsilon^i)} = [(1-\varepsilon^2)^{1/2}\delta^j_i + \eta^j_{\cdot ik}\varepsilon^k]\frac{\partial}{\partial \varepsilon^j} \tag{29}$$

$$\tilde{X}^L_{(V^i)} = R^k_i\left(\frac{\partial}{\partial V^k} + mA_k\Xi\right)$$

$$\tilde{X}^L_{(\zeta)} = \Xi$$

Galilei Θ,

$$\Theta = -m\mathbf{A}\cdot d\mathbf{V} - \tfrac{1}{2}m\mathbf{V}^2\, dB + \frac{d\zeta}{i\zeta} \tag{30}$$

(28) and (30) were extensively used in Aldaya and de Azcárraga (1982).

The observation of (19), (26) makes explicitly manifest that there is *no* biunivocal correspondence between the group $P\otimes U(1)$ and the Ξ component Θ of the left-invariant 1-form defined on it, which depends on the coboundary selected to write explicitly the group law. Nevertheless, as Θ is determined by the coboundary (Section 2) the structure of $\mathscr{P}\otimes U(1)\to^{\pi}\mathscr{P}$ as a principal fiber bundle *with connection* Θ is completely determined. This result is a consequence of the following theorem on invariant connections (Kobayashi and Nobizu, 1963):

Theorem. Let G be a connected Lie group and H a closed subgroup; let $\mathscr{G}$ and $\mathscr{H}$ be the Lie algebras of G and H, respectively.

(a) If there is a subspace $\mathscr{M}$ of $\mathscr{G}$ such that $\mathscr{G}=\mathscr{M}\oplus\mathscr{H}$ and $\mathrm{ad}(H)\mathscr{M}=\mathscr{M}$, then the $\mathscr{H}$ component Θ of the canonical 1-form θ on G with respect the decomposition $\mathscr{G}=\mathscr{M}\oplus\mathscr{H}$ defines a connection on the bundle $G(G/H, H)$ which is left invariant under the left translations of G.

(b) Conversely, any connection in $G(G/H, H)$ invariant under the left translations of G determines such a decomposition $\mathscr{G}=\mathscr{M}\oplus\mathscr{H}$ and is obtained in the manner described in (a).

(c) The curvature 2-form cur Θ of the invariant connection Θ is given by

$$\Omega(X, Y) = [X, Y]|_{\mathscr{H}} \equiv -\Theta([X, Y])$$

for any arbitrary left-invariant vector fields X, Y on G belonging to $\mathscr{M}$.

In our case, $G \equiv \mathcal{P} \otimes U(1)$, $H \equiv U(1)$. If the function δ generating the coboundary fulfills (11), $\mathcal{M}$ is a subalgebra (that of $\mathcal{P}$) and the curvature is flat [(11′)]; when (11) is not satisfied, then the curvature is given by (c) above.

Table I collects a set of examples for which δ_c is undefined including those discussed in this section.

4. QUANTIZATION ON A GROUP MANIFOLD AND CONTRACTION

The above analysis is of special relevance in connection with the formulation of a quantum dynamics on a group manifold. Strictly speaking, the quantization formalism based on a symmetry group breaks down when it is applied to the relativistic case because the Poincaré group does not admit (quantum) extensions by $U(1)$ other than direct product ones. Thus, one is led to consider infinite Lie groups, graded Lie groups, or perhaps even both. (The $N=2$ super-Poincaré extended by a central charge, for example, has already been studied (Aldaya and de Azcárraga, 1983)). In this last section we analyze in which way the considerations of Section 3 could constitute a partial way out of the no $U(1)$-extension problem for the quantization of relativistic boson systems.

Following a previous work (Aldaya and de Azcárraga, 1982), we start directly with the form generated by the coboundary (25), which have (28) and (30) as their Galilean limits. The election of an extension characterized by a $\delta(g)$ which does not fulfill (11) may be justified by the requirement that the *group law* of $\mathcal{P} \otimes U(1)$ (or, what it is equivalent, the connection form Θ) has to give as nonrelativistic limit the quantum Galilei group $\tilde{G}_{(m)}$. The characteristic module of Θ, $\{\tilde{X} / i_{\tilde{X}}\Theta = i_{\tilde{X}}\, d\Theta = 0\}$ is generated by $\tilde{X}^L_{(A^0)}$, $\tilde{X}^L_{(\varepsilon i)}$ and this set, completed with $\tilde{X}^L_{(\mathbf{A})}$ generate an *horizontal subalgebra* which defines the *full polarization*. The Hilbert space of the wave functions is then characterized by the conditions

$$\Xi \cdot \Psi = i\Psi, \qquad \Psi = \Psi(A^\mu, \boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \zeta) \tag{31}$$

[$U(1)$ function] which implies $\Psi = \zeta\Phi(A^\mu, \boldsymbol{\varepsilon}, \boldsymbol{\alpha})$ plus

$$\tilde{X}^L_{(\boldsymbol{\varepsilon})} \cdot \Psi = 0, \qquad \tilde{X}^L_{(\mathbf{A})} \cdot \Psi = 0, \qquad \tilde{X}^L_{(A^0)} \cdot \Psi = 0 \tag{32}$$

the first two of which remove the $\mathbf{A}$ (space) and $\boldsymbol{\varepsilon}$ dependence and the third gives

$$\Phi = e^{-i(p^0 - mc)x^0}_{*} \varphi(\mathbf{p}) \tag{33}$$

Table I

$\delta(g)$	$-mcA^0$	$-A^{\mu}p_{\mu}$	$\mathbf{A}\cdot\mathbf{p}-mcA^0$	$-A^0p^0$
$\xi(g',g)$	$-2mc\boldsymbol{\alpha}'^2A^0$ $-2mc(1+\alpha'^2)^{1/2}\boldsymbol{\alpha}'\cdot R'\mathbf{A}$	$-A'^{\mu}(\Lambda'p-p')_{\mu}$	$\left[\frac{\mathbf{p}'}{mc}A^0+R'\mathbf{A}+\frac{(\mathbf{p}'\cdot R'\mathbf{A})\mathbf{p}'}{mc(p_0'+mc)}\right]$ $\times\left[\gamma\mathbf{p}'+R'\mathbf{p}+\frac{(\mathbf{p}'\cdot R'\mathbf{p})\mathbf{p}'}{mc(p_0'+mc)}\right]$ $-(p_0'-mc)A^0-\mathbf{p}'\cdot R'\mathbf{A}$	$-\frac{p_0'p_0}{mc}\left[A'^0+\gamma'A^0+\frac{\mathbf{p}'\cdot R'\mathbf{A}}{mc}\right]$ $-\frac{\mathbf{p}'\cdot R'\mathbf{p}}{mc}\left[A'^0+\gamma'A^0+\frac{\mathbf{p}'\cdot R'\mathbf{A}}{mc}\right]$ $+A'^0p_0'+A^0p_0$
$\xi_c(g',g)$	$-m(\mathbf{V}'\cdot R'\mathbf{A}+\frac{1}{2}\mathbf{V}'^2B)$	$m[\mathbf{A}'\cdot R'\mathbf{V}-B'(\frac{1}{2}\mathbf{V}^2+\mathbf{V}'\cdot R'\mathbf{V})]$	$m[\mathbf{A}'\cdot R'\mathbf{V}+B(\mathbf{V}'\cdot R'\mathbf{V}+\frac{1}{2}\mathbf{V}'^2)]$	$-m[\mathbf{V}'\cdot R'\mathbf{A}+B(\mathbf{V}'\cdot R'\mathbf{V}+\mathbf{V}'^2)$ $+B(\mathbf{V}'\cdot R'\mathbf{V}+\frac{1}{2}\mathbf{V}^2)]$
$X^{\zeta}_{(A^0)}$	$-2mc\alpha^2$	0	$2mc\alpha^2(1+2\alpha^2)$	$-4mc\alpha^2(1+\alpha^2)$
$X^{\zeta}_{(A^k)}$	$-2mc(1+\alpha^2)^{1/2}\alpha_jR^j_k$	0	$4mc\alpha^2(1+\alpha^2)^{1/2}\alpha_jR^j_k$	$-2mc(1+2\alpha^2)(1+\alpha^2)^{1/2}\alpha_jR^j_k$
$X^{\zeta}(\varepsilon^i)$	0	0	0	0
$X^{\zeta}_{(\alpha^i)}$	0	$-2mc[A^02(1+\alpha^2)^{1/2}\alpha_kR^k_i$ $-A_j(R^j_i+2\alpha^j\alpha_kR^k_i)]$	$2mcA_k[R^k_i+2\alpha^k\alpha_jR^j_i]$	$-4mc(1+\alpha^2)^{1/2}A^0\alpha_jR^j_i$
$X^{\zeta}_{c(B)}$	$-\frac{1}{2}m\mathbf{V}^2$	0	$\frac{1}{2}m\mathbf{V}^2$	$-m\mathbf{V}^2$
$X^{\zeta}_{c(A^k)}$	$-mR^i_kV_i$	0	0	$-mV_jR^j_k$
$X^{\zeta}_{c(\varepsilon^i)}$	0	0	0	0
$X^{\zeta}_{c(V^i)}$	0	$m[A_k-BV_k]R^k_i$	$mA_kR^k_i$	$-mBV_jR^j_k$
Θ	$\mathbf{p}\cdot d\mathbf{A}-(p_0-mc)\,dA^0+\frac{d\zeta}{i\zeta}$	$A^{\mu}\,dp_{\mu}+\frac{d\zeta}{i\zeta}$	$-\mathbf{A}\cdot d\mathbf{p}-(p^0-mc)\,dA^0+\frac{d\zeta}{i\zeta}$	$\mathbf{p}\cdot d\mathbf{A}+A^0\,dp_0+\frac{d\zeta}{i\zeta}$
Θ_c	$m\mathbf{V}\cdot d\mathbf{A}-\frac{1}{2}m\mathbf{V}^2\,dB+\frac{d\zeta}{i\zeta}$	$-m\mathbf{A}\cdot d\mathbf{V}+Bd(\frac{1}{2}m\mathbf{V}^2)+\frac{d\zeta}{i\zeta}$	$-m\mathbf{A}\cdot d\mathbf{V}-\frac{1}{2}m\mathbf{V}^2\,dB+\frac{d\zeta}{i\zeta}$	$m\mathbf{V}\cdot d\mathbf{A}+Bd(\frac{1}{2}m\mathbf{V}^2)+\frac{d\zeta}{i\zeta}$

where we have rewritten $A^0 = x^0$ and used $\mathbf{p}$ instead of $\boldsymbol{\alpha}$ [(14)]. Thus, the equation of motion $(\tilde{X}^L_{(A^0)} \cdot \Psi = 0)$ is

$$i\frac{\partial\varphi(t, \mathbf{p})}{\partial t} = [c(m^2c^2 + \mathbf{p}^2)^{1/2} - mc^2]\varphi(t, \mathbf{p}) \tag{34}$$

which leads to the Schrödinger equation in the nonrelativistic limit. This was to be expected; to obtain the nonrelativistic limit from the Klein–Gordon equation

$$i\frac{\partial\varphi(t, \mathbf{p})}{\partial t} = c(m^2c^2 + \mathbf{p}^2)^{1/2}\varphi(t, p) \tag{35}$$

the rest energy mc^2 has to be substracted by means of a redefinition $\varphi \to e^{imcx^0}\varphi$. This allows the limit to be performed since mc is not defined for $c \to \infty$. It should be remarked that this change of variables $\zeta' = e^{imcx^0}\zeta$, $x^{0'} = x^0$ modifies (27) by an additional exact form $d(-mcA^0)$,

$$\Theta' = -\mathbf{A} \cdot d\mathbf{p} - p_0\, dA^0 + \frac{d\zeta}{i\zeta} \tag{36}$$

which obviously does not alter the symplectic structure of the phase space. However, the same change in the *group law* leads to the explicitly trivial composition law and to $\Theta = d\zeta/i\zeta$ (Section 2). Moreover, the 1-form (36) which leads to the Klein–Gordon equation can only be derived as the dual of a set of vector fields that, although they reproduce a Poincaré action, do not give by integration the Poincaré Lie group but an orbit in a bigger group. (Aldaya and de Azcárraga, 1985.) Indeed, it may be checked that the addition of the terms $mc(1+2\alpha^2)\Xi$, $2mc(1+\alpha^2)^{1/2}R^j_k\alpha_j\Xi$ to the expressions of $\tilde{X}^L_{(A^0)}$ and $\tilde{X}^L_{(A^k)}$, respectively, in (26) is sufficient to derive both (35) and (36). However, the new algebra, although fulfills the same commutation relations as the algebra of (26) does not correspond to the action of a Lie group on itself written in canonical coordinates.

To conclude we remark that the use of coboundaries defining nonflat connections, which as shown above solves the problem of defining the nonrelativistic limit in a geometric manner, breaks nevertheless the canonicity of the quantization method at this stage in the sense that Nature has to choose between isomorphic groups. On the contrary, the use of symmetry groups as abstract symmetry structures (as in the Galilean case where the quantum group $\tilde{G}_{(m)}$ is *characterized* by a cohomology *class* of G) leads us to the need of substituting more ample structures for the Poincaré group (Aldaya and de Azcárraga, 1982, 1983, 1985).

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